

# SPIN STRUCTURES ON FLAT MANIFOLDS

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**ABSTRACT.** We present an algorithmic approach to the problem of existence of spin structures on flat manifolds. We apply our method in the cases of flat manifolds of dimensions 5 and 6.

## 1. INTRODUCTION

Let  $\Gamma$  be an  $n$  dimensional crystallographic group, i.e. a discrete and cocompact subgroup of the group  $E(n) = O(n) \ltimes \mathbb{R}^n$  of isometries of the Euclidean space  $\mathbb{R}^n$ . By Bieberbach theorems (see [1, 2, 3]),  $\Gamma$  fits into short exact sequence

$$(1) \quad 0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where  $\mathbb{Z}^n$  is a maximal abelian normal subgroup of  $\Gamma$  and  $G$  is a finite group, so called holonomy group of  $\Gamma$ . When in addition  $\Gamma$  is torsionfree, then  $\Gamma$  is called a Bieberbach group. In this case the orbit space  $\mathbb{R}^n/\Gamma$  is a flat manifold, i.e. closed connected Riemannian manifold with sectional curvature equal to zero.

Existence of a spin structure on a manifold  $X$  allows us to define on  $X$  a Dirac operator. Every oriented flat manifold of dimension less or equal to 3 admits a spin structure. In dimension 4, 24 out of 27 flat manifolds has spin structures (see [17]). In this paper we present an algorithm to determine existence of a spin structure on flat manifolds and present some facts concerning spin structures on flat manifolds of dimensions 5 and 6.

Section 2 recalls some basic definitions and introduces a notation concerning Clifford algebras. The main goal of Section 3 is to present a more flexible form of a Pfäffle criterion of existence of spin structures on flat manifolds. The key tool in looking for spin structures on a flat manifold is the restriction of its holonomy representation to the Sylow 2-subgroup of the holonomy group. In Section 4 we show that this restriction can be realized in a very convenient form and in Section 5 we show its usage in the criterion mentioned above. The algorithm for determining spin structures on flat manifolds is presented in Section 6 and is followed by example of its usage for a 5-dimensional flat manifold. The last section presents some facts about spin structures for 5 and 6 dimensional manifolds.

## 2. CLIFFORD ALGEBRAS AND SPIN GROUPS

**Definition 1.** Let  $n \in \mathbb{N}$ . The *Clifford algebra*  $C_n$  is a real associative algebra with one, generated by elements  $e_1, \dots, e_n$ , which satisfy relations:

$$\forall_{1 \leq i < j \leq n} e_i^2 = -1 \wedge e_i e_j = -e_j e_i.$$

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*Remark 1.* We have the following  $\mathbb{R}$ -algebras isomorphisms:

$$C_0 \cong \mathbb{R}, \quad C_1 \cong \mathbb{C}, \quad C_2 \cong \mathbb{H}.$$

*Remark 2.* We may view  $\mathbb{R}^n := \text{span}\{e_1, \dots, e_n\}$  as a vector subspace of  $C_n$ , for  $n \in \mathbb{N}$ .

**Definition 2** (Three involutions). Let  $n \in \mathbb{N}$ . We have the following involutions of  $C_n$ :

- $*$ :  $C_n \rightarrow C_n$ , defined on the basis of (the vector space)  $C_n$  by

$$\forall 1 \leq i_1 < i_2 < \dots < i_k \leq n \quad (e_{i_1} \dots e_{i_k})^* = e_{i_k} \dots e_{i_1};$$

- $'$ :  $C_n \rightarrow C_n$ , defined on the generators of (the algebra)  $C_n$  by

$$\forall 1 \leq i \leq n \quad e_i' = -e_i.$$

- $\bar{\phantom{x}}$ :  $C_n \rightarrow C_n$  – the composition of the previous involutions

$$\forall a \in C_n \quad \bar{a} = (a')^*.$$

We are now ready to define the spin groups as subgroups of unit groups in the clifford algebras:

**Definition 3.**

$$\forall n \in \mathbb{N} \quad \text{Spin}(n) := \{x \in C_n \mid x' = x \wedge x\bar{x} = 1\}.$$

**Proposition 1** ([18, Prop. 6.1, page 86], [8, page 16]). Let  $n \in \mathbb{N}$ . The map  $\lambda_n: \text{Spin}(n) \rightarrow \text{SO}(n)$ , defined by

$$\forall x \in \text{Spin}(n) \quad \forall v \in \mathbb{R}^n \quad \lambda_n(x)v = xv\bar{x}$$

is a continuous group epimorphism with kernel equal to  $\{\pm 1\}$ . Moreover for  $n \geq 3$   $\text{Spin}(n)$  is simply connected and  $\lambda_n$  is the universal covering of  $\text{SO}(n)$ .

### 3. SPIN STRUCTURES ON (FLAT) MANIFOLDS

**Definition 4.** Let  $X$  be an orientable closed manifold of dimension  $n$ . Let  $Q$  be its principal  $\text{SO}(n)$ -tangent bundle. A *spin structure* on  $X$  is a pair  $(P, \Lambda)$ , such that  $P$  is a principal  $\text{Spin}(n)$ -bundle over  $X$  and  $\Lambda: P \rightarrow Q$  is a 2-fold covering for which the following diagram commutes:

$$\begin{array}{ccc} P \times \text{Spin}(n) & \longrightarrow & P \\ \downarrow \Lambda \times \lambda_n & & \downarrow \Lambda \\ Q \times \text{SO}(n) & \longrightarrow & Q \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad X$$

where the maps in rows are defined by the action of the groups  $\text{Spin}(n)$  and  $\text{SO}(n)$  on the principal bundles  $P$  and  $Q$  respectively.

**Proposition 2** ([8, page 40]). An orientable closed manifold  $X$  has a spin structure if and only if its second Stiefel-Whitney class vanishes:

$$w_2(X) = 0.$$

Moreover in this case spin structures on  $X$  are classified by  $H^1(X, \mathbb{Z}_2)$ .

By the following proposition determining spin structures on flat manifolds becomes purely algebraic.

**Proposition 3** ([16, Proposition 3.2]). *Let  $X$  be an  $n$ -dimensional orientable flat manifold with fundamental group  $\Gamma \subset E(n)$ . Then the set of spin structures on  $X$  is in bijection with the set of homomorphisms of the form  $\varepsilon: \Gamma \rightarrow \text{Spin}(n)$  for which the following diagram commutes:*

$$(2) \quad \begin{array}{ccc} & & \text{Spin}(n) \\ & \nearrow \varepsilon & \downarrow \lambda_n \\ \Gamma & \xrightarrow{r} & \text{SO}(n) \end{array}$$

where  $r: \Gamma \rightarrow \text{SO}(n)$  takes the rotational part of every element of  $\Gamma$ :

$$\forall_{(A,a) \in \Gamma \subset \text{SO}(n) \ltimes \mathbb{R}^n} r(A, a) = A.$$

*Remark 3.* By a little abuse of notation we will call  $r: \Gamma \rightarrow \text{SO}(n)$  the *holonomy representation* of  $\Gamma$ , since in our case a representation equivalent to the holonomy representation of the manifold  $\mathbb{R}^n/\Gamma$  is the identity map  $id: r(\Gamma) \rightarrow r(\Gamma) \subset \text{SO}(n)$ .

Now let  $X = \mathbb{R}^n/\Gamma$  be an orientable flat manifold with fundamental group  $\Gamma \subset E(n)$ . The group  $\Gamma$  is finitely presented. Let

$$\Gamma = \langle S \mid R \rangle$$

be its presentation with the set of generators  $S$  closed under taking inversions ( $S^{-1} = S$ ) and the set of relations  $R$ , both finite sets. A map  $\varepsilon: S \rightarrow \text{Spin}(n)$  can be extended to a homomorphism  $\varepsilon: \Gamma \rightarrow \text{Spin}(n)$  if and only if it preserves the relations of  $\Gamma$ :

$$\forall_{r_1, \dots, r_l \in S} r_1 \cdot \dots \cdot r_l \in R \Rightarrow \varepsilon(r_1) \cdot \dots \cdot \varepsilon(r_l) = 1.$$

Moreover, since  $\ker \lambda_n = \{\pm 1\}$ , in order to get commutativity of the diagram (2) we must have

$$\forall_{\gamma \in \Gamma} \forall_{x \in \text{Spin}(n)} r(\gamma) = \lambda_n(x) \Rightarrow \varepsilon(\gamma) = x \vee \varepsilon(\gamma) = -x.$$

Hence to check if we can construct a homomorphism  $\varepsilon$  then for every generator  $s \in S$  it is enough to find an element  $x \in \text{Spin}(n)$  such that

$$r(s) = \lambda_n(x)$$

and check which combinations of signs of those elements of  $\text{Spin}(n)$  preserve relations of  $\Gamma$ .

In general it is not easy task to find preimages  $\lambda_n^{-1}(g)$  of an element  $g \in \text{SO}(n)$ . The following proposition allows us to search for such finite subgroups of  $\text{SO}(n)$  which are easier to work with.

**Proposition 4** ([10, Proposition 2.1]). *Let  $n \in \mathbb{N}$  and  $\Gamma_1, \Gamma_2 \subset E(n)$  be isomorphic Bieberbach groups. Then the set of spin structures on the flat manifold  $\mathbb{R}^n/\Gamma_1$  is in bijection with the set of spin structures on the flat manifold  $\mathbb{R}^n/\Gamma_2$ .*

The bijection in the above proposition is given as follows. Let  $\Gamma_1, \Gamma_2$  be Bieberbach groups as above. Let  $r_i: \Gamma_i \rightarrow \text{SO}(n)$  be the holonomy representations and let  $G_i = r_i(\Gamma_i)$  be the holonomy groups of  $\Gamma_i$ , for  $i = 1, 2$ . If  $\varepsilon_1: \Gamma_1 \rightarrow \text{Spin}(n)$  defines a

spin structure on  $\mathbb{R}^n/\Gamma_1$  then the corresponding homomorphism  $\varepsilon_2: \Gamma_2 \rightarrow \text{Spin}(n)$  fits into the following commutative diagram

$$(3) \quad \begin{array}{ccc} \varepsilon_1(\Gamma_1) & \xrightarrow{\alpha} & \text{Spin}(n) \\ \varepsilon_1 \nearrow & \downarrow \lambda_n & \searrow \varepsilon_2 \\ \Gamma_1 & \xrightarrow{\Phi} & \Gamma_2 \\ r_1 \nearrow & \downarrow \varphi & \searrow r_2 \\ G_1 & \xrightarrow{\varphi} & \text{SO}(n) \end{array}$$

where  $\Phi: \Gamma_1 \rightarrow \Gamma_2$  is the isomorphism,  $\varphi$  is the homomorphism induced by  $\Phi$  and  $\alpha$  is induced by  $\varphi$ .

**Corollary 1.** *Let  $\Gamma \subset E(n)$  be a Bieberbach group with holonomy representation  $r: \Gamma \rightarrow \text{SO}(n)$  and holonomy group  $G = r(\Gamma)$ . The set of spin structures on the flat manifold  $\mathbb{R}^n/\Gamma$  is in bijection with the set of homomorphisms of the form  $\varepsilon: \Gamma \rightarrow \text{Spin}(n)$  for which the following diagram commutes:*

$$(4) \quad \begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon} & \text{Spin}(n) \\ r \downarrow & & \downarrow \lambda_n \\ G & \xrightarrow{\varphi} & \text{SO}(n) \end{array}$$

where  $\varphi: G \rightarrow \text{SO}(n)$  is a representation of  $G$  equivalent to the identity map  $\text{id}: G \rightarrow G \subset \text{SO}(n)$ .

#### 4. FLAT MANIFOLDS WITH 2-GROUP HOLONOMY

**Proposition 5** ([5, Proposition 1]). *Let  $n \in \mathbb{N}$ . Let  $\Gamma \in E(n)$  be a Bieberbach group with holonomy representation  $r: \Gamma \rightarrow \text{SO}(n)$  and holonomy group  $G = r(\Gamma)$ . Let  $S \subset G$  be a 2-Sylow subgroup of  $G$ . Then the flat manifold  $\mathbb{R}^n/\Gamma$  admits a spin structure if and only if  $\mathbb{R}^n/r^{-1}(S)$  admits one.*

By Corollary 1 in the process of determining existence of spin structures on a flat manifold we can choose any subgroup of  $\text{SO}(n)$  which is conjugated in  $\text{GL}(n, \mathbb{R})$  to its holonomy group. By the above proposition it is enough to look on 2-subgroups of  $\text{SO}(n)$ . In this section we will show that for every 2-group in  $\text{SO}(n)$  we can find its conjugate  $G$  in such a way that  $\lambda_n^{-1}(G)$  is easy to compute.

*Remark 4.* The extension (1) defines the *integral holonomy representation*  $\rho: G \rightarrow \text{GL}(n, \mathbb{Z})$  defined by the conjugations in  $\Gamma$ :

$$\forall g \in G \forall z \in \mathbb{Z}^n \rho_g(z) = \gamma z \gamma^{-1},$$

where  $\gamma$  is an element of  $\Gamma$  such that  $r(\gamma) = g$ . This representation is  $\mathbb{R}$ -equivalent to the "identity representation"  $\text{id}: G \rightarrow G \subset \text{SO}(n)$ .

**Theorem 6** ([6, Theorem 1.10]). *Let  $G$  be a finite  $p$ -group and let  $\varphi: G \rightarrow \text{GL}(m, \mathbb{Q})$  be an irreducible representation over  $\mathbb{Q}$ . Then either  $\varphi$  is induced from a representation of a subgroup of index  $p$  or  $[G : \ker \varphi] \leq p$ .*

By an induction argument we immediately get

**Corollary 2.** *Every irreducible rational representation of 2-group is induced from a rational representation of degree 1.*

Now let's take a closer look on a matrix representation of a 2-group  $G$

$$\varphi: G \rightarrow \mathrm{GL}(n, \mathbb{Q}).$$

By Corollary 2 we may assume that

$$\varphi = \mathrm{ind}_H^G \tau$$

where  $H$  is a subgroup of  $G$  and  $\tau: H \rightarrow \mathbb{Q}^*$  is a representation of  $H$  of degree 1. Since  $\tau(H) \subset \{\pm 1\}$ , we get that every element of  $\varphi(G)$  is an orthogonal integral matrix and  $\varphi$  is if the form

$$\varphi: G \rightarrow \mathrm{O}(n, \mathbb{Z}) := \mathrm{O}(n) \cap \mathrm{GL}(n, \mathbb{Z}).$$

Now if a 2-group  $G \subset \mathrm{SO}(n)$  is a holonomy group of a Bieberbach group  $\Gamma \subset E(n)$  then by Corollary 1 the set of spin structures of the manifold  $\mathbb{R}^n/\Gamma$  is in bijection with the set of homomorphisms  $\varepsilon: \Gamma \rightarrow \mathrm{Spin}(n)$  which make the following diagram commute

$$(5) \quad \begin{array}{ccc} \Gamma & \xrightarrow{\varepsilon} & \mathrm{Spin}(n) \\ \downarrow r & & \downarrow \lambda_n \\ G & \xrightarrow{\varphi} & \mathrm{SO}(n, \mathbb{Z}) \end{array}$$

where  $\mathrm{SO}(n, \mathbb{Z}) = \mathrm{SL}(n, \mathbb{Z}) \cap \mathrm{SO}(n)$ . This seems to be a minor change in comparison to Corollary 1, but it simplifies a lot the problem of determining preimages of  $\lambda_n$ .

## 5. SPECIAL ORTHOGONAL GROUP OVER INTEGERS

In this section we will show how to determine the preimage of any element of the group  $\mathrm{SO}(n, \mathbb{Z})$  under the homomorphism  $\lambda_n$  for  $n \geq 2$ . Recall that in this case  $\ker \lambda_n = \{\pm 1\}$  so calculation of one element in the preimage immediately gives us the other one.

The group  $\mathrm{O}(n, \mathbb{Z})$  fits into the following exact sequence

$$1 \longrightarrow N \longrightarrow \mathrm{O}(n, \mathbb{Z}) \longrightarrow S_n \longrightarrow 1,$$

where  $S_n$  is the symmetric group on  $n$  letters and  $N \subset \mathrm{O}(n, \mathbb{Z})$  is the group of diagonal matrices with  $\pm 1$  on the diagonal. The sequence splits and the splitting homomorphism sends a permutation  $\sigma \in S_n$  to its permutation matrix  $P_\sigma \in \mathrm{O}(n, \mathbb{Z})$ .

Now let  $X \in \mathrm{SO}(n, \mathbb{Z})$  be an integral orthogonal matrix. There exist inversions  $\sigma_1, \dots, \sigma_k \in S_n$  and a diagonal integral matrix  $D \in N$  such that

$$X = P_{\sigma_1} \cdots P_{\sigma_k} \cdot D.$$

Unfortunately matrices of inversions have determinant equal to  $-1$  and they don't belong to  $\mathrm{SO}(n, \mathbb{Z})$ . A little modification changes this fact. Let  $(p \ q) \in S_n$  be an inversion with  $p < q$ . Define matrix  $P'_{(p \ q)} \in \mathrm{SO}(n, \mathbb{Z})$  as follows:

$$P'_{(p \ q)} = \mathrm{diag}(\underbrace{1, \dots, 1}_{p-1}, -1, 1, \dots, 1) \cdot P_{(p \ q)}.$$

We get that

$$(6) \quad X = P'_{\sigma_1} \cdots P'_{\sigma_k} \cdot D'$$

where  $D' \in N$  but this time all the factors  $P'_{\sigma_1}, \dots, P'_{\sigma_k}, D'$  in decomposition of  $X$  have determinant 1 and hence they are elements of  $\mathrm{SO}(n, \mathbb{Z})$ . In order to determine  $\lambda_n^{-1}(X)$  it is enough to calculate the preimages of its factors:

- (1) If  $D' \in N$  is a matrix with  $-1$  on the diagonal entries  $n_1, \dots, n_l$  ( $l$  even) then

$$(7) \quad \lambda_n(\pm e_{n_1} \cdots e_{n_l}) = D'.$$

- (2) If  $(p\ q) \in S_n$  is an inversion with  $p < q$  then

$$(8) \quad \lambda_n\left(\pm \frac{1 + e_p e_q}{\sqrt{2}}\right) = P'_{(p\ q)}.$$

## 6. NOTES ABOUT THE ALGORITHM

Let  $n \geq 2$ . Assume that  $\Gamma' \subset E(n)$  is a Bieberbach group with holonomy representation  $r: \Gamma' \rightarrow \mathrm{SO}(n)$  and that  $\Gamma'$  fits into the following short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{i} \Gamma' \xrightarrow{r} G' \longrightarrow 1.$$

The following steps will determine existence of spin structures on the flat manifold  $\mathbb{R}^n/\Gamma'$ .

**Step 1:** Determine a Sylow 2-subgroup  $G$  of  $G'$  and its preimage  $\Gamma = r^{-1}(G)$  in  $\Gamma'$ . We get an extension

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{r} G \longrightarrow 1$$

where  $r$  is in fact a restriction  $r|_{\Gamma}$ .

**Step 2:** Determine a representation  $\varphi: G \rightarrow \mathrm{SO}(n, \mathbb{Z})$  of a 2-group  $G$  which is  $\mathbb{R}$ -equivalent to the identity representation  $id_G: G \rightarrow G \subset \mathrm{SO}(n)$ . Note that it may be helpful to build a list of all  $\mathbb{Q}$ -irreducible integral and orthogonal representations of  $G$ . Since we are in characteristic zero the character theory is very useful in determining which of those are subrepresentations of  $id_G$ .

**Step 3:** Fix a generating set  $\{g_1, \dots, g_s\}$  of  $G$ . For every  $1 \leq i \leq s$  decompose  $\varphi(g_i)$  as in (6) and then, using formulas (7) and (8) determine  $x_i \in \mathrm{Spin}(n)$  such that

$$\lambda_n(x_i) = \varphi(g_i).$$

**Step 4:** Determine the integral holonomy representation

$$\varrho: G \rightarrow \mathrm{GL}(n, \mathbb{Z}).$$

Denote by  $\varrho_{i,j}(g) \in \mathbb{Z}$  the entry in the  $i$ -th row and  $j$ -th column of the matrix  $\varrho(g) \in \mathrm{GL}(n, \mathbb{Z})$  where  $1 \leq i, j \leq n, g \in G$ . It is worth to notice that CARAT uses the integral holonomy representation to store crystallographic groups as a subgroup of  $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Q}^n$  with translation lattice being always  $\mathbb{Z}^n$ . In this form the projection on the first coordinate defines the integral holonomy representation. Note that this is not a constraint in any way, since  $\varrho, id_G$  and  $\varphi$  are all  $\mathbb{R}$ -equivalent.

**Step 5:** Let  $a_1, \dots, a_n \in \Gamma$  be the images of the generators of  $\mathbb{Z}^n$  in  $\Gamma$ . Let  $\gamma_1, \dots, \gamma_s$  be elements of  $\Gamma$  such that

$$\forall_{1 \leq i \leq s} r(\gamma_i) = g_i.$$

By [11, Proposition 1, page 139]

$$\Gamma = \langle a_1, \dots, a_n, \gamma_1, \dots, \gamma_s \rangle.$$

Note that if we have a homomorphism  $\varepsilon: \Gamma \rightarrow \text{Spin}(n)$  such that  $\lambda_n \varepsilon = r$  then

$$\varepsilon(a_i) \in \{\pm 1\} \wedge \varepsilon(\gamma_j) \in \{\pm x_i\}$$

for all  $1 \leq i \leq n, 1 \leq j \leq s$ . Now for every possible value of a function  $\varepsilon$  on the generators of  $\Gamma$  we have to check whether we can extend it to a homomorphism of groups, i.e. we have to check whether the images preserve relations of the generators of  $\Gamma$  which are of three types:

- (1) Relations which come from the monomorphism  $\mathbb{Z}^n \rightarrow \Gamma$  are the commutator relations and they are automatically satisfied, since all the generators  $a_1, \dots, a_n$  are mapped to the center of  $\text{Spin}(n)$ .
- (2) Relations which come from the action of  $G$  on  $\mathbb{Z}^n$ . Let  $1 \leq i \leq n$  and  $1 \leq j \leq s$ . Using the holonomy representation  $\varrho$  we get the following relation in  $\Gamma$ :

$$\gamma_j a_i \gamma_j^{-1} = a_1^{\varrho_{1i}(g_j)} \dots a_n^{\varrho_{ni}(g_j)}.$$

The corresponding relation in  $\text{Spin}(n)$  should be as follows

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \dots \varepsilon(a_n)^{\varrho_{ni}(g_j)} = \varepsilon(\gamma_j) \varepsilon(a_i) \varepsilon(\gamma_j)^{-1} = \varepsilon(a_i)$$

since  $\varepsilon(a_i) = \pm 1$ . From the same reason the above equation may be written as

$$\varepsilon(a_1)^{\varrho_{1i}(g_j)} \dots \varepsilon(a_n)^{\varrho_{ni}(g_j)} \varepsilon(a_i) = 1.$$

- (3) Relations which come from relations of  $G$ . Let

$$g_{i_1} \dots g_{i_k}$$

be a relator of  $G$  (you can skip inverses since  $G$  is finite). Then

$$\gamma_{i_1} \dots \gamma_{i_k} = a_1^{\alpha_1} \dots a_n^{\alpha_n}$$

for some  $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ . The resulting relation in  $\text{Spin}(n)$  is

$$\varepsilon(\gamma_{i_1}) \dots \varepsilon(\gamma_{i_k}) = \varepsilon(a_1)^{\alpha_1} \dots \varepsilon(a_n)^{\alpha_n},$$

which is equivalent to

$$\varepsilon(\gamma_{i_1}) \dots \varepsilon(\gamma_{i_k}) \varepsilon(a_1)^{\alpha_1} \dots \varepsilon(a_n)^{\alpha_n} = 1.$$

## 7. EXAMPLE

Let  $\Gamma'$  be a Bieberbach group generated by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1/3 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the matrices of the form

$$(9) \quad a_i = \begin{bmatrix} I & e_i \\ 0 & 2 \end{bmatrix}$$

where  $I$  is the identity matrix of degree 5 and vectors  $e_i, i = 1, \dots, 5$  are generators of  $\mathbb{Z}^5$ . The group is denoted in CARAT by min.134.1.2.2. The holonomy group  $G'$  of  $\Gamma'$  is isomorphic to the symmetric group  $S_4$  and so its 2-Sylow subgroup  $G$  is isomorphic to the dihedral group  $D_8$ . If  $r: \Gamma' \rightarrow \text{SO}(5)$  is the holonomy representation, then the preimage  $\Gamma = r^{-1}(G)$  is generated by the matrices  $a_1, \dots, a_5$  and the following ones:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using character theory we get that a faithful representation  $\varphi: G \rightarrow \text{SO}(5, \mathbb{Z})$ ,  $\mathbb{R}$ -equivalent to  $id_G$ , may be defined by

$$r(A) \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, r(B) \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

We get that

$$\lambda_5(\pm e_2 e_4) = \varphi r(A), \quad \lambda_5\left(\pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}\right) = \varphi r(B).$$

For a map  $\epsilon: \Gamma \rightarrow \text{Spin}(5)$  to be a homomorphism, we have to have the following relations:

(1) The action of  $G$  on  $\mathbb{Z}^5$ :

$$(10) \quad \begin{cases} \varepsilon(a_2)\varepsilon(a_3) = 1, \\ \varepsilon(a_2)\varepsilon(a_4)\varepsilon(a_5) = 1. \end{cases}$$

(2) The relations from  $G$ . We have

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$(AB)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$



We get that the following relations should be satisfied in  $\text{Spin}(5)$

$$(11) \quad \begin{cases} \varepsilon(A)^2 \varepsilon(a_2) \varepsilon(a_3) \varepsilon(a_4) \varepsilon(a_5) = 1 \\ \varepsilon(B)^4 \varepsilon(a_4) \varepsilon(a_5) = 1 \\ (\varepsilon(A) \varepsilon(B))^2 \varepsilon(a_2) \varepsilon(a_4) = 1 \end{cases}$$

From (10) and (11) we get the following conditions on values of  $\varepsilon$ :

$$\begin{cases} \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4) \varepsilon(a_5) = \varepsilon(A)^2 = \varepsilon(B)^4 \\ \varepsilon(a_5) = (\varepsilon(A) \varepsilon(B))^2 \end{cases}$$

Note that both values of  $\varepsilon(a_1)$  are allowed. Recall that

$$\varepsilon(A) = \pm e_2 e_4 \text{ and } \varepsilon(B) = \pm \frac{e_2 e_3 (1 + e_5 e_4)}{\sqrt{2}}.$$

Since for any of the above values we have  $\varepsilon(A)^2 = \varepsilon(B)^4 = (\varepsilon(A) \varepsilon(B))^2 = -1$ , hence we get 8 spin structures on  $\mathbb{R}^5/\Gamma$  and there exists a spin structure on  $\mathbb{R}^5/\Gamma'$ . Moreover, since  $H^1(\mathbb{R}^5/\Gamma', \mathbb{Z}_2) = \mathbb{Z}_2^2$ , we get exactly four spin structures on the former manifold.

## 8. SOME STATISTICS

Recall that CARAT represent any  $n$ -dimensional Bieberbach group  $\Gamma$  as subgroup of  $\text{GL}(n, \mathbb{Z}) \ltimes \mathbb{Q}^n$ . In this representation the maximal normal abelian subgroup equals  $\mathbb{Z}^n$ , the holonomy group  $G \cong \Gamma/\mathbb{Z}^n$  is a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  and the integral holonomy representation is just the inclusion map to  $\text{GL}(n, \mathbb{Z})$ . By the  $\mathbb{Z}$ -class and the  $\mathbb{Q}$ -class of finite subgroup of  $\text{GL}(n, \mathbb{Z})$  we will denote the conjugacy class of the group in  $\text{GL}(n, \mathbb{Z})$  and  $\text{GL}(n, \mathbb{Q})$  respectively.

Necessary condition for a Bieberbach group to have a spin structure is to be orientable. This property is fully determined by the  $\mathbb{Q}$ -class of the holonomy group. Note that existence of spin structures is – in contrast to orientation – determined only by the isomorphism class of a Bieberbach group (see Remark 5). Table 2 shows the number of  $\mathbb{Q}$ -classes, the number of  $\mathbb{Q}$ -classes which determine orientable flat manifolds and the number of  $\mathbb{Q}$ -classes for which exists flat manifold with a spin structure in dimensions 5 and 6.

Table 3 shows the number of all flat manifolds, orientable flat manifolds and flat manifolds which admit a spin structure in dimensions 5 and 6. Because of their number, in Table 1 we list all Bieberbach groups with spin structures of dimension 5. The data for both dimensions 5 and 6 can be downloaded from the WWW page [12].

*Remark 5.* From the paper [17] the following facts hold for flat manifolds in dimension 4:

- (1) Existence of a spin structure does not depend on the  $\mathbb{Q}$ -class of the integral holonomy representation of an orientable flat manifold.
- (2) Existence of a spin structure is determined by the  $\mathbb{Z}$ -class of the integral holonomy representation of an orientable flat manifold.

By [10, Example 3.3] and [14, Theorem 3.2] we know that the former fact does not hold in dimension 6. The calculations give 5-dimensional examples – for each of the following  $\mathbb{Z}$ -classes of finite subgroups of  $\text{GL}(5, \mathbb{Z})$  there exist Bieberbach groups with holonomy group in the class, with and without spin structures:

$\Gamma'$	$G'$	$r^{-1}(G)$	#S	$\Gamma'$	$G'$	$r^{-1}(G)$	#S
min.58.1.1.0	1	min.58.1.1.0	32	min.85.1.1.44	$D_8$	min.85.1.1.44	8
min.59.1.1.1	$C_2$	min.59.1.1.1	32	min.85.1.1.45	$D_8$	min.85.1.1.45	8
min.62.1.1.1	$C_2$	min.62.1.1.1	32	min.85.1.1.46	$D_8$	min.85.1.1.46	8
min.62.1.2.1	$C_2$	min.62.1.2.1	16	min.85.1.3.19	$D_8$	min.85.1.3.19	4
min.62.1.3.1	$C_2$	min.62.1.3.1	8	min.85.1.3.22	$D_8$	min.85.1.3.22	8
min.65.1.1.7	$(C_2)^2$	min.65.1.1.7	16	min.86.1.13.5	$D_8$	min.86.1.13.5	4
min.66.1.1.11	$(C_2)^2$	min.66.1.1.11	16	min.86.1.13.6	$D_8$	min.86.1.13.6	8
min.66.1.3.11	$(C_2)^2$	min.66.1.3.11	8	min.86.1.13.7	$D_8$	min.86.1.13.7	4
min.70.1.1.20	$(C_2)^2$	min.70.1.1.20	32	min.90.1.10.3	$C_4 \times C_2$	min.90.1.10.3	4
min.70.1.1.22	$(C_2)^2$	min.70.1.1.22	16	min.98.1.3.12	$C_4 \times C_2$	min.98.1.3.12	4
min.70.1.1.28	$(C_2)^2$	min.70.1.1.28	16	min.101.1.1.1	$C_3$	min.58.1.1.0	2
min.70.1.1.30	$(C_2)^2$	min.70.1.1.30	16	min.104.1.1.1	$C_3$	min.58.1.1.0	8
min.70.1.14.1	$(C_2)^2$	min.70.1.14.1	4	min.104.1.2.1	$C_3$	min.58.1.1.0	8
min.70.1.15.19	$(C_2)^2$	min.70.1.15.19	16	min.106.1.1.1	$C_6$	min.62.1.1.1	8
min.70.1.15.5	$(C_2)^2$	min.70.1.15.5	8	min.107.1.1.2	$S_3$	min.62.1.2.1	8
min.70.1.1.76	$(C_2)^2$	min.70.1.1.76	16	min.107.1.2.1	$S_3$	min.62.1.3.1	4
min.70.1.1.77	$(C_2)^2$	min.70.1.1.77	16	min.107.2.1.2	$S_3$	min.62.1.2.1	8
min.70.1.1.94	$(C_2)^2$	min.70.1.1.94	16	min.107.2.2.1	$S_3$	min.62.1.3.1	4
min.70.1.2.25	$(C_2)^2$	min.70.1.2.25	8	min.107.2.3.2	$S_3$	min.62.1.2.1	8
min.70.1.2.9	$(C_2)^2$	min.70.1.2.9	16	min.107.2.4.1	$S_3$	min.62.1.3.1	4
min.70.1.3.11	$(C_2)^2$	min.70.1.3.11	8	min.110.1.1.1	$C_6$	min.59.1.1.1	8
min.70.1.3.7	$(C_2)^2$	min.70.1.3.7	16	min.110.1.3.1	$C_6$	min.59.1.1.1	8
min.70.1.4.10	$(C_2)^2$	min.70.1.4.10	8	min.123.1.1.1	$C_{12}$	min.79.1.1.1	4
min.70.1.4.11	$(C_2)^2$	min.70.1.4.11	8	min.124.1.1.1	$C_{12}$	min.81.1.1.1	4
min.70.1.4.7	$(C_2)^2$	min.70.1.4.7	16	min.129.1.1.1	$C_6$	min.62.1.1.1	2
min.70.1.4.9	$(C_2)^2$	min.70.1.4.9	8	min.129.1.2.1	$C_6$	min.62.1.3.1	2
min.70.1.6.3	$(C_2)^2$	min.70.1.6.3	8	min.130.1.1.12	$(C_3)^2$	min.58.1.1.0	2
min.70.1.7.13	$(C_2)^2$	min.70.1.7.13	8	min.130.1.1.37	$(C_3)^2$	min.58.1.1.0	2
min.70.1.7.15	$(C_2)^2$	min.70.1.7.15	8	min.130.1.3.10	$(C_3)^2$	min.58.1.1.0	2
min.71.1.1.362	$(C_2)^3$	min.71.1.1.362	16	min.131.1.2.3	$A_4$	min.70.1.15.19	4
min.71.1.1.371	$(C_2)^3$	min.71.1.1.371	8	min.131.2.1.3	$A_4$	min.70.1.1.76	4
min.71.1.1.373	$(C_2)^3$	min.71.1.1.373	16	min.132.1.2.3	$C_2 \times A_4$	min.71.1.25.95	4
min.71.1.1.375	$(C_2)^3$	min.71.1.1.375	8	min.132.2.1.6	$C_2 \times A_4$	min.71.1.1.373	4
min.71.1.1.378	$(C_2)^3$	min.71.1.1.378	8	min.134.1.2.2	$S_4$	min.86.1.13.6	4
min.71.1.1.382	$(C_2)^3$	min.71.1.1.382	8	min.154.1.1.1	$C_{12}$	min.75.1.1.1	2
min.71.1.25.95	$(C_2)^3$	min.71.1.25.95	16	min.164.1.1.1	$C_5$	min.58.1.1.0	2
min.75.1.1.1	$C_4$	min.75.1.1.1	8	group.240.2.1.11	$D_8$	group.240.2.1.11	8
min.79.1.1.1	$C_4$	min.79.1.1.1	16	group.326.1.1.1	$C_6$	min.59.1.1.1	2
min.79.1.2.2	$C_4$	min.79.1.2.2	8	group.341.1.1.1	$C_6$	min.62.1.1.1	8
min.81.1.1.1	$C_4$	min.81.1.1.1	16	group.361.1.1.21	$D_{12}$	min.70.1.3.7	8
min.81.1.3.1	$C_4$	min.81.1.3.1	8	group.361.1.1.22	$D_{12}$	min.70.1.3.11	4
min.81.1.6.1	$C_4$	min.81.1.6.1	8	group.541.1.1.10	$C_6 \times C_3$	min.62.1.1.1	2
min.85.1.1.41	$D_8$	min.85.1.1.41	16	group.994.1.1.1	$C_{10}$	min.59.1.1.1	2
min.85.1.1.42	$D_8$	min.85.1.1.42	8				

TABLE 1. Spin structures in dimension 5.  $\Gamma'$  – the name of the Bieberbach group,  $G'$  – isomorphism type of the holonomy group of  $\Gamma'$ ,  $r^{-1}(G)$  – the preimage of the Sylow 2-subgroup of  $G$ , #S – the number of spin structures on  $\mathbb{R}^5/\Gamma'$

Dim	#QC	#OQC	#SQC
5	95	41	34
6	397	106	85

TABLE 2. Number of all  $\mathbb{Q}$ -classes (QC), orientable  $\mathbb{Q}$ -classes (OQC) and the ones for which exist spin manifolds (SQC) in dimensions 5 and 6

Dim	#FM	#OFM	#SFM
5	1060	174	87
6	38746	3314	717

TABLE 3. Number of all, oriented and spin flat manifolds in dimensions 5 and 6

min.66.1.1, min.66.1.3, min.70.1.1, min.70.1.15, min.70.1.2, min.70.1.3,  
min.70.1.7, min.71.1.1, min.71.1.25, min.85.1.3, group.361.1.1.

Those  $\mathbb{Z}$ -classes belong to the following 5  $\mathbb{Q}$ -classes:

min.66, min.70, min.71, min.85.1.3, group.361.

Moreover there are 95  $\mathbb{Z}$ -classes of finite subgroups of  $\mathrm{GL}(6, \mathbb{Z})$ , collected in 34  $\mathbb{Q}$ -classes, for which we can find examples of Bieberbach groups with and without spin structures.

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